Chapter # 5

ENHANCED CONCEPTUALIZATION OF LEARNING ALGEBRA THROUGH SCIENTIFIC-ORIENTED TEACHING

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ABSTRACT
The purpose of this project is to analyze how the scientific concept of inverse operations can be used as a "bottom-up" approach for teaching mathematical operations with whole and rational numbers. The primary aim of the study is to provide support for teachers in the teaching of scientific mathematical concepts to students of algebra. The theoretical approaches applied in this project are the theories of mathematical structures, especially the theory of inverse semigroups (Abelian groups), and relational thinking, in a comparison with instrumental thinking. The methodological approach uses Vygotsky's doctrine of scientific concepts. The presentation of the analytical findings expands on previous research on additive and multiplicative inverses in teaching and conceptual learning of algebra. The findings illustrate how connections to mathematical structures and scientific concepts, such as the concept of inverse, lead to an understanding of algebraic operation rules for whole and rational numbers.

Keywords: inverse operations, teaching and learning algebra, conceptual understanding, operations with negative and rational numbers, conceptual meaning and algebraic rules.

1. INTRODUCTION

International research on teaching and learning mathematics is continuously evolving. The reconstruction and development of school curricula and international discourse on mathematics education require teachers to constantly evaluate their view of teaching and choice of mathematical content to support students' sustainable learning. The latest psychological research highlights the importance of conceptual learning in order to situate meaning sense for the conceptual understanding of mathematical concepts, formulas, and methods (Schwartz & Martin, 2004; Hall, Wieckert, & Wright, 2008). This basically means that students form an understanding of mathematics in relation to the invisible mathematical context behind concepts, formulas, and methods, which helps them reconstruct and reproduce mathematical knowledge and skills. The crucial component of conceptual mathematics learning is the link between students' learning and scientific concepts in a "bottom-up" didactic teaching approach that should lead to individual conceptual growth in the students' progress in relation to mathematical premises (Vygotsky, 1986). Conceptual learning also provides a deeper understanding of mathematical procedures, as well as the significance of concepts and formulas.

Researchers agree that the quality of teaching and learning of mathematics depends on the teacher's ability to teach the subject, which plays a central role in their students' ability to acquire mathematical skills. The findings of international empirical and theoretical studies imply that teachers' knowledge of mathematics and their current teaching strategies do not provide sufficient support for their students' development in the subject (Ball, Hill, & Bass, 2005). What kind of teaching content can provide the optimal interplay between a student's cognitive ability to learn mathematics and their conceptual development in the subject? How should mathematical content be structured to ensure that students understand core mathematical concepts and methods? International research currently has a limited capacity to address such these questions.

According to Vygotsky (1986): "The only good kind of instruction is that which marches ahead of development and leads it" (p. 188). A key to teachers' acquisition of mathematical knowledge for teaching is that they understand the basic ideas of mathematics and didactic tools for teaching.

2. BACKGROUND ALGEBRA AND CONCEPTUAL UNDERSTANDING

Researchers agree that teaching basic algebra such as whole and rational numbers is no easy task because it requires theoretical knowledge. It demands theoretical knowledge about the content. According to Kieran (2004), building of mathematical structures and understanding of number sense in arithmetic creates a conceptual bridge to algebra and other areas of mathematics. Knowledge of number theory and mathematical operations is key to ensuring continuity in the learning of algebra (Kozulin, Gindis, Ageyev, & Miller, 2003). From this point of view, a teacher needs a good overview of (and good insight into) the relevant content in operations, as well as opportunities to identify the multifaceted perceptions of the operations and the logical continuity of algebraic thinking. At the same time, it is important to understand "bottom up" and "breaking down" didactic approaches in the teaching of algebra, as well as the nature of school algebra in relation to mathematics as a science.

The conceptual transition and continuity in students' learning of algebra has been analytically described by Kaput (2008) and in relation to students' learning of rational numbers and their operations in a study by Karlsson and Kilborn (2024a). Researchers agree that teaching algebra effectively is a difficult process and impossible to carry out with only procedural knowledge. More specifically, the generalization of algebra cannot take place without the conceptual generalization of both algebraic concepts and number concepts, including whole numbers, rational numbers, and algebraic operations. At the same time, researchers emphasize that students have difficulties mastering algebraic concepts and the concept of rational numbers (fractions) (Alajmi, 2012; Lee, Choy, & Mizzi, 2021; Berggren, 2022).

It is important that as early as the first school year, students understand that all arithmetic rules are based on algebra. Hence, the best way to teach algebra is to introduce students to the laws of arithmetic from grade 1. At this grade 1, it is also important for students to become familiar with the relationship between the properties of numbers and operations (Karlsson & Kilborn, 2015). One example is the relationship between addition and subtraction (e.g., $7 - 4 = x \Leftrightarrow 4 + x = 7$), which leads students to a conceptual understanding of the subtraction operation "take away". Another example is that $7 - (-5)$ can be illustrated on the number line as the distance from (-5) to 7, that is, the number 12 (12 steps between two numbers). The understanding of inverse as an algebraic concept is key to students' learning of addition and inverse subtraction of whole numbers, multiplication, and inverse division of rational numbers (Karlsson & Kilborn, 2024b). In more general terms, the concept of inverse plays an important role in students' learning of content such as the square of a number (and its inverse, the square root) and the exponential function (and its inverse, the logarithmic function). Students already know everyday inverses like “heads and tails” or “passed and failed,” and false inverses like day and night or clever and stupid. In fact, the concept of inverse should be successively introduced and actively used in teaching from primary to lower and upper secondary school in order for students to learn mathematical ideas and their implications in the form of the conceptual contexts of rules and formulas.

In line with this, research should be expanded to address the question of students’ conceptual learning of algebraic concepts through the teaching of fundamental algebra in combination with didactic methods for teaching algebra. This chapter introduces the analytical findings from our research, which illustrate a beneficial approach to mathematical structures, such as the relation between the scientific concept of inverse and the teaching and learning of the conceptual meaning of operations of whole (negative) numbers and rational numbers.

The purpose of this study is to analyze how the algebraic concepts of additive and multiplicative inverse operations can be used as a “bottom-up” approach for teaching algebraic operations with whole and rational numbers. The primary aim of this analytical research is to provide support for the teaching of algebra in primary and secondary school, with a focus on students’ conceptual learning of algebra. This research builds on previous research on additive and multiplicative inverses in teaching and learning arithmetic and algebra (Karlsson & Kilborn, 2023; 2024b) and enhances algebra instruction.

3. THEORETICAL PERSPECTIVES ON MATHEMATICAL STRUCTURE AND MATHEMATICAL THINKING

Mathematical structures are crucial to students’ knowledge of mathematics and teachers’ ability to teach the subject. In this context, mathematical structures stimulate deeper mathematical thinking and help students to engage in the learning process more effectively (Mason, Stephens, & Watson, 2009). The analysis of mathematical structures can also elucidate the connection between “conceptual and procedural approaches to teaching and learning mathematics” (p. 10). In line with this, it is relevant to express ideas about mathematical structures related to teachers’ didactic knowledge of mathematics and how this knowledge can support their students’ learning. According to Mason et al. (2009), a mathematical structure is defined through “the identification of general properties which are instantiated in particular situations as relationships between elements” (p. 10). This underlines two important aspects of the definition of a mathematical structure. The first has to do with the mathematical meaning of general/essential properties, and the second with the “association” between elements. These elements are mathematical objects or concepts, for example, number concepts, algebraic concepts, and related operations. In this context, the operations are related within elements of natural numbers, whole numbers, and rational numbers. In a conceptual sense, the relationship between elements is a connection between these number concepts and the operations that include them, as well as an extension of these operations to algebraic elements/concepts. Understanding mathematical structures as relationships between essential properties within a mathematical structure (mathematical concept), the relationship between different elements and “mastering procedures” facilitates the development of contextual and conceptual meaning in an individual’s mathematical knowledge. The understanding of mathematical structure can result in more effective learning focused on the re-construction and extension of existing mathematical knowledge and the construction of new mathematical knowledge (Mason et al., 2009).

The theoretical issues surrounding mathematical structure clearly indicate which kind of mathematical knowledge makes sense in teaching and learning. In general terms, “making sense” means understanding mathematical structures and the relationships between them in an algebraic context (Kieran, Pang, Schifter, & Fong, 2016). Discussions about mathematical structures in instruction lead to theoretical considerations about the phenomenon of mathematical (algebraic) thinking and contextual sense. Mathematical thinking is important for learners’ analysis of mathematical (algebraic) structures in order to understand content and apply this knowledge to conceptual and procedural learning. Researchers have used a framework for mathematical thinking related to domains such as relational thinking (Skemp, 1976; Starr, Vendetti, & Bunge, 2018) and instrumental thinking (Mason et al., 2009). Skemp (1976) describes the distinction between two different types of understanding (thinking) as relational understanding and instrumental understanding. He defines the concept of understanding as “knowing both what to do and why” (p. 2). Relational thinking/understanding encompasses such issues as what to do and why. Answering the above requires the ability to generalize mathematical structures/concepts according to their essential properties and to make conceptual and procedural connections with other mathematical structures/concepts. Relational thinking includes understanding the properties of numbers, mathematical operations with numbers, and properties for inverse operations, as well as the ability to handle number sentences. In terms of pre-service learning for the teaching of relational thinking, it is important to support the conceptualization of mathematical operations, restructure operations related to transition from one number range to another, as well as teach students the skills they need to generalize mathematical operations from the use of natural numbers and rational numbers to algebraic symbols (Molina & Mason, 2009).

Definitions of Abelian groups have been described by van der Waerden (1971). Mathematical structures, such as the Abelian groups for addition and multiplication for the teaching and learning of algebra in terms of fraction ideas, have been described in previous research (Karlsson & Kilborn, 2024a; 2024b). Mathematical structures refer to students’ ability to develop essential skills in algebra, understanding of the relationships between natural numbers and rational numbers, as well as algebraic rules for addition, including the concept of additive and multiplicative inverse operations. With regard to teaching algebra, research also highlights the importance of Abelian groups as a specific form of mathematical knowledge for effective instruction and the development of conceptual knowledge. According to these researchers (Karlsson & Kilborn, 2023), algebra is a difficult subject to teach and learn. Further, it is also difficult to generalize algebraic concepts. Hence, it is important to investigate how students develop an understanding of general algebraic concepts from a scientific perspective.

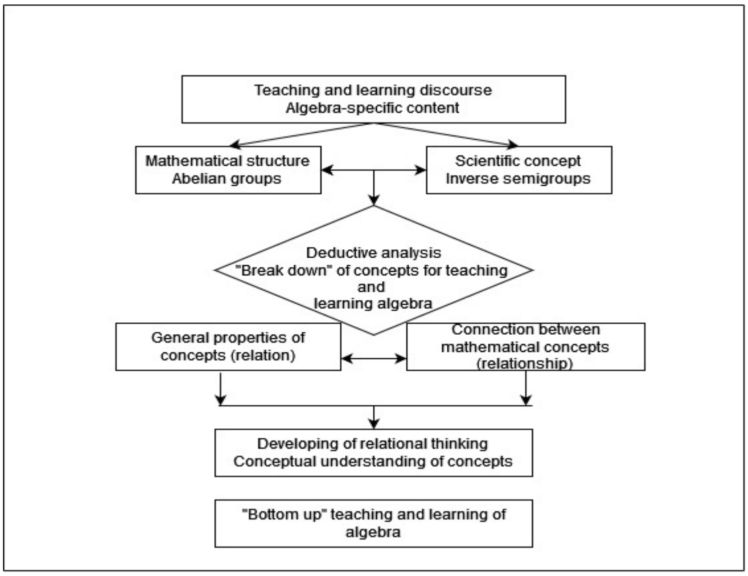
4. METHODOLOGY

The methodology of this study is rooted in socio-constructivism. This methodological approach uses Vygotsky’s doctrine of scientific concepts (Karpov, 2003). According to Vygotsky’s doctrine, mathematics and algebra are learned in two ways: (1) The first way is when students use “spontaneous” concepts related to the conceptualization and generalization of everyday experiences and knowledge. (2) The second way is grounded in “scientific” concepts and students’ learning of mathematics (algebra) through the generalization and the systematic accumulation of human scientific knowledge, which is closely related to mathematical nature, ideas, and the historical development of mathematics as a science with a focus on what lies behind different formulas, concepts, and methods. These two kinds of knowledge create the conditions and opportunities for conceptual learning and mathematical reasoning and thinking. It is crucial to understand that spontaneous concepts are prerequisites for the construction of scientific concepts.

4.1. Scientific Knowledge and Concepts

A closer investigation of what students should learn (learning outcomes) turns the lens on classroom instruction and how content should be taught in relation to student' learning. In the context of scientific knowledge and theoretical views on scientific (theoretical) knowledge and its significance for mathematics learning in the conceptual sense (Karpov, 2003), this study relies on the core theories of socio-constructivism, specifically the crucial role of theoretical thinking and learning. In this approach, theoretical thinking is applied in the investigation of abstract mathematics as a school subject. The application of theoretical perspectives to mathematical structures is key to the development of mathematical thinking in teachers, as well as for their understanding of the same process among their students. It is also crucial to emphasize that theoretical-analytical learning effectively maximizes the opportunities for abstract thinking and the generalization of content and concepts. The question of how teaching content can be broken down into logical sequences that learners can adapt and absorb is a key aspect of mathematical knowledge for teaching. Against this background, and in terms of the mathematical preparation of teaching for teachers, content knowledge should be theorized in a manner that underscores the clear interplay between theoretical knowledge and its impact on the teaching of mathematics. The theoretical approach of this study focuses on how scientific knowledge and concepts lead to reflections on why theoretical concepts should be taught as a basis for learning of mathematics, specifically algebra (Bakirov & Turgunbaev, 2019). The systematized theoretical framework of the study is presented in Figure 1.

Figure 1.
Systemized theoretical framework: Teaching and learning of algebra.



4.2. Analysis Track

To track the construction of the deductive analysis of the inverse semigroups for the teaching of algebra, we have utilized a conceptual chart of theoretical starting points in combination with a methodology about mathematical structure/scientific concepts in the teaching and learning of mathematics. The methodology of this study is also grounded in a systematic analysis of the essential properties of the concept of inverse. The analytical construction of the teaching and learning discourse, as well as the findings related to the teaching of algebra with a focus on whole numbers, rational numbers, and algebraic operations, is deeply grounded in the researcher's awareness and understanding of the school's mathematics-algebra discourse.

5. RESEARCH FINDINGS, SCIENTIFIC CONCEPT, AND TEACHING

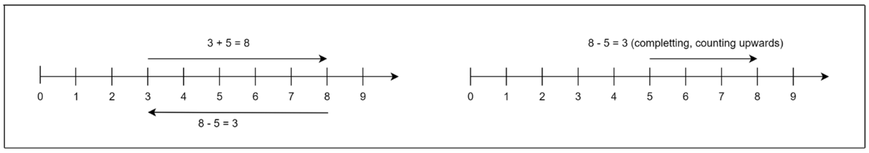
The findings of the study are presented with a focus on the relational and conceptual components of additive and multiplicative inverse concerning the teaching of whole and rational numbers and operations with these numbers.

5.1. Finding 1. Addition of Whole Numbers and their Inverse, Subtraction

The additive inverse (van der Waerden, 1971) can be illustrated by the natural number 5 and its additive inverse (-5) as a negative number, where the sum of the numbers 5 and (-5) is zero. On the number line, this can be expressed as being mirrored around the neutral element 0. This illustrates how negative numbers are inverses of their positive, natural counterparts.

On the number line, adding 5 can be expressed as an arrow five units in length, pointing to the right; subtracting 5 can be expressed as an arrow five units in length pointing to the left. The addition equation 3 + 5 = 8 and the subtraction equation 8 – 5 = 3 are illustrated on the number line in Figure 2.

Figure 2.
Addition and inverse operation subtraction.



The subtraction equation 8 – 5 = x can also be carried out as the inverse of the open addition equation 5 + x = 8 by *completing on the number line,* that is, counting up from 5 to 8 in 3 steps. This is shown on the number line in Figure 2 (counting upwards). If the rule regarding the sum of two whole numbers is applied ($b + (-b) = 0$), it is easy to understand how to carry out the subtraction equation $a - (-b)$. We start by subtracting $a - b$ and then add ($b + (-b)$). By using the commutative law of addition, we find that $a - b = a + (-b)$. This means that $a - b = a +$ (the inverse of b). However, as this is true for all whole numbers, it also means that $a - (-b)$ equals $a +$ (the inverse of $(-b)$) $= a + b$. The relational thinking about the addition of whole numbers and their inverse, subtraction can be extended to rational numbers. The overall conceptual goal is for students to understand basic algebra to explain the following operations, as well as to build an understanding of the essential properties of whole numbers in order to understand algebraic concepts and methods. These are given by the Abelian semigroup for addition, where:

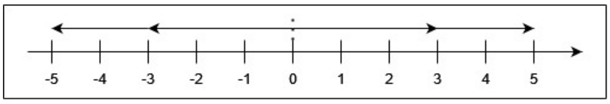
1. the same arithmetic laws apply to whole numbers as to natural numbers.
2. *negative numbers* are inverses of those natural numbers.
3. subtraction is the inverse operation of addition which leads to $a - b = a + (-b)$ or at a deeper level, $a - (-b) = a +$ (*the inverse of* $(-b)) = a + b$.
4. in order to be able to follow algebraic reasoning, for example, $a + (-b) = a + (b - b) + (-b) = a - b + b + (-b)) = a - b$.

5.2. Finding 2. Negative Numbers and Additive Inverse Operations

Case 1. Addition of negative numbers

Negative numbers can also be broken down into terms. In the same way as 5 = 3 + 2, the inverse applies as (-5) = (-3) + (-2); see Figure 3.

Figure 3.
Negative numbers as inverses.



Against this background, it is easy to understand the addition of whole numbers, for example:
(1) 7 + (-3) = 4 + 3 + (-3) = 4 because 3 + (-3) = 0, which should be compared to 7 – 3 = 4; and
(2) 4 + (-9) = 4 + (-4) + (-5) = (-5) since 4 + (-4) = 0, which should be compared to 4 – 9 = (-5).

Case 2. Subtraction of negative numbers

Applying the concept of inverses and the fact that $(b + (-b)) = 0$ leads to $a - b = a + (b + (-b)) - b = a + (-b) + (b - b) = a + (-b)$. This means that $a - b = a +$ (*inverse number of* b) = if *a and b* are whole numbers. As a conceptual consequence of inverse operation, this method enables an understanding of subtraction of negative numbers. For example:
(1) 6 – 2 = 6 + (the inverse number of 2) = 6 + (-2); and
(2) 6 – (- 2) = 6 + (the inverse number of (- 2)) = 6 + 2.

The subtraction of negative numbers is confusing for many students. One reason for this is that they often learn algebraic rules by rote, such as “Two equal signs make a plus sign,” although the two signs for “minus” are very different. By learning inverse algebraic operation, it is possible for students to understand the relational and conceptual meanings of algebraic rules.

Case 3. Multiplication with negative number(s)

To examine what it means to multiply two whole numbers, we can start with $a \cdot (-b)$, where $a > 0$. We then apply the distributive law: $a \cdot b + a \cdot (-b) = a \cdot (b + (-b)) = 0$. This means that $a \cdot b$ and $a \cdot (-b)$ are inverse numbers and, consequently, that $a \cdot (-b) = (-a\ b)$ the same way that $(-a) \cdot b = (-a \cdot b)$.

How do we interpret the multiplication $(-a) \cdot (-b)$ of two negative numbers? By once again using the distributive law, we find that $(-a) \cdot b + (-a) \cdot (-b) = (-a) \cdot (b + (-b)) = 0$, which means that $(-a) \cdot (-b)$ is the inverse of $(-a) \cdot b$, whose inverse is $a \cdot b$. Consequently, $(-a) \cdot (-b) = ab$. In sum, the finding is that the product of two negative numbers is a positive number. By understanding multiplicative inverse, it is possible for students to understand the relational and conceptual meanings of algebraic rules with a focus on the multiplication of negative numbers.

5.3. Finding 5. Rational Numbers and Inverse Operations

Concerning multiplication, for every a in the group (provided that $a \neq 0$), there is an element $\frac{1}{a}$, such that $a \cdot \frac{1}{a} = 1$. In this case, $\frac{1}{a}$ is the (multiplicative) inverse of a and vice versa (van der Waerden, 1971).

Case 1. Multiplication of rational numbers

The analytical findings of inverse are related to *the multiplication of two rational numbers*. To begin with, let us multiply two basic fractions, $\frac{1}{a}$ and $\frac{1}{b}$, and then multiply their product by (a · b). We then apply the commutative and associative laws:

$a\,b \cdot (\frac{1}{a} \cdot \frac{1}{b}) = (a \cdot \frac{1}{a}) \cdot (b \cdot \frac{1}{b}) = 1 \cdot 1 = 1$. Since the product $a\,b \cdot (\frac{1}{a} \cdot \frac{1}{b}) = 1$, the inverse of $(\frac{1}{a} \cdot \frac{1}{b})$ is $a\,b$. However, the inverse of $a\,b$ is $\frac{1}{a \cdot b}$. Consequently, $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{a \cdot b}$. This means that $\frac{a}{b} \cdot \frac{c}{d} = a \cdot \frac{1}{b} \cdot c \cdot \frac{1}{d} = ac \cdot \frac{1}{bd} = \frac{ac}{bd}$. Consequently, $\frac{a}{b}$ is the inverse of $\frac{b}{a}$. Case 1 illustrates the conceptual meaning of the formula for the multiplication of rational numbers.

Case 2. Division of rational numbers

In case 2, the inverse operation for division is defined in a way that can also be applied to the division of rational numbers. Such a definition is that $a \div b = a \cdot \frac{1}{b}$. This means that the division of two rational numbers can be interpreted as $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$ and more explicitly, this means that $\frac{2}{7} \div \frac{1}{3} = \frac{2}{7} \cdot \text{inv}\,(\frac{1}{3}) = \frac{2}{7} \cdot 3 = \frac{6}{7}$ and $\frac{2}{5} \div \frac{3}{7} = \frac{2}{5} \cdot \frac{7}{3}$. Against this background, students with only a basic understanding of algebra can conceptually understand the formula for *the division of two fractions*.

In order to carry out division where $\frac{a}{b} / \frac{c}{d}$, the concept of inverse is used. Since $\frac{c}{d} \cdot \frac{d}{c} = 1$, then $\frac{d}{c}$ is the inverse of $\frac{c}{d}$. The definition of $\frac{a}{b}$ is a ·(inverse of b). This logically leads to $\frac{a}{b} / \frac{c}{d} = \frac{a}{b} \cdot$ (inverse of $\frac{c}{d}$) $= \frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c}$. This is a conceptual explanation of the formula for the division of rational numbers.

This is an interesting point for relational thinking. Since $\frac{a}{2} \div \frac{a}{2} = 1$ and $\frac{2}{a} \cdot \frac{a}{2} = 1$ independent of a, everything depends on the conceptual meaning of the division formula as a result of an inverse operation multiplication.

6. FUTURE RESEARCH DIRECTIONS

Future research directions could be to develop more effective instructions with a focus on mathematical structures and scientific concepts, as well as the impact on students' conceptual learning. It is also important to investigate the conceptual potential of Abelian groups and the implementation of the concept in school algebra in a structured and logical manner. An important future research direction is to conduct empirical studies that investigate teacher educators' practices in algebra-oriented learning, as well as to incorporate the theoretical findings of the present study, namely, the teaching model for negative and rational numbers as a means to enhance students' understanding and the effectiveness of mathematics education.

7. DISCUSSION AND CONCLUSION

This study analyses how the algebraic concept of additive and multiplicative inverse operations can be used as a "bottom-up" approach for teaching mathematical operations with whole and rational numbers. The findings show that mathematical structures and the concept of inverse operations of rational numbers can provide teachers with systematic conceptual support when planning instruction. This can help them teach algebra effectively, specifically whole and rational numbers and operations that include these numbers, in order to develop their students' relational thinking and understanding of algebraic rules and formulas (Skemp, 1976). At the same time, the theoretical framework of this research (Figure 1) can provide teachers with a clearer understanding of the methods of working from the "bottom up" and "breaking mathematical concepts down" as two different approaches (inductive and deductive) for teaching and learning. The analytical findings show a clear connection to how the mathematical structures/scientific concepts of inverse operations for addition and multiplication can be implemented and handled to promote the conceptual understanding of algebra, while employing teaching-based practice. This suggests that all students should master operations with negative and rational numbers instead of focusing on the rote memorization of rules and formulas. However, the ability to make decisions about teaching quality and approaches to the teaching and learning of algebra depend on the teaching content and interplay between teaching content and intended learning outcomes (Molina & Mason, 2009; Kieran et al., 2016; Karlsson & Kilborn, 2023; 2024a; 2024b).

These analytical results can enhance the ability of teachers' and students to grasp these mathematical operations and inform the approaches used in the teaching and learning of such operations. This study also illustrates how mathematical structures and scientific concepts can be implemented in school mathematics for teaching and learning purposes. The implementation of innovative teaching approaches linked to a deeper scientific understanding can provide a conceptual device and the ability to grasp the meaning of algebraic concepts that students have not yet mastered, concepts that are important for sensemaking in their learning.

The analytical results of the study provide analytical frameworks for subsequent empirical studies with a focus on teaching and students' learning of algebra. The results also contribute to the effectiveness of mathematics teaching and learning in the conceptual sense and provide new knowledge on approaches to the teaching and learning of mathematical structures and scientific concepts.

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Background in brief: Natalia Karlsson is an associate professor of mathematics at Södertörn University, Sweden. Her qualifications include a master's degree in mathematics, a master's degree in education, and a PhD in mathematics and physics. Her research areas are mathematics education and applied mathematics. Her areas of expertise in the field of mathematics education are mathematical content in teaching and students' learning of mathematics. Her research has a focus on developing theoretical and methodological approaches to teaching, crucial mathematical content, and the variation of and the relationships between the teaching and learning of mathematics.

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Background in brief: Wiggo Kilborn is a researcher in the innovative PUMP project, which initiated a new methodology and a new technique for studying the teaching process in Sweden. He has also contributed to the development of the didactics of mathematics in Sweden. He has applied his experience to developing courses for teacher training and supporting the development of research and education in several countries, such as Portugal, South Africa, Mozambique, Zimbabwe, Tanzania, and Guinea-Bissau.